

HOSSAM GHANEM

(10) 7.5 Integration Of Inverse Trigonometric Functions (C)

Integration

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right)$$

$$\int \frac{1}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right)$$

Example 1

Evaluate the integral

$$\int \frac{1}{(x^2 + 1) \tan^{-1} x} dx$$

16 November 2004
6 March 1997

Solution

$$t = \tan^{-1} x$$

$$dt = \frac{1}{(x^2 + 1)} dx$$

$$I = \int \frac{1}{t} dt = \ln|t| + c = \ln|\tan^{-1} x| + c$$

Example 2

Evaluate the integral

$$\int \frac{x^2}{\sqrt{1 - x^6}} dx$$

20 November 2006 A

Solution

$$t = x^3$$

$$dt = 3x^2 dx$$

$$\frac{1}{3} dt = x^2 dx$$

$$I = \frac{1}{3} \int \frac{1}{\sqrt{1 - t^2}} dt = \frac{1}{3} \sin^{-1} t + c = \frac{1}{3} \sin^{-1} x^3 + c$$

Example 3

Evaluate the following integral

$$\int \frac{\sin x}{\sqrt{1 + \sin^2 x}} dx$$

22 July 2007

Solution

$$I = \int \frac{\sin x}{\sqrt{1 + \sin^2 x}} dx = \int \frac{\sin x}{\sqrt{1 + 1 - \cos^2 x}} dx = \int \frac{\sin x}{\sqrt{2 - \cos^2 x}} dx$$

$$t = \cos x$$

$$dt = -\sin x dx$$

$$I = \int \frac{1}{\sqrt{2 - t^2}} dt = -\sin^{-1} \left(\frac{t}{\sqrt{2}} \right) + c = -\sin^{-1} \left(\frac{\cos x}{\sqrt{2}} \right) + c$$

Example 4

Evaluate the following integral

$$\int \frac{1}{(1 + \cos^2 x) \csc x} dx. \quad 25 \text{ April } 2008$$

Solution

$$I = \int \frac{1}{(1 + \cos^2 x) \csc x} dx. = - \int \frac{1}{1 + t^2} dt = - \tan^{-1} t + c = - \tan^{-1}(\cos x) + c$$

$t = \cos x \quad dt = -\sin x dx \quad -dt = \frac{1}{\csc x} dx$

Example 5

Evaluate the following integral

$$\int \frac{dx}{x((\log_3 x)^2 + 1)} \quad 5 \text{ October } 1996$$

Solution

$$I = \int \frac{dx}{x((\log_3 x)^2 + 1)} = \ln 3 \int \frac{1}{t^2 + 1} dt = \ln 3 \tan^{-1} t + c = \ln 3 \tan^{-1}(\log_3 x) + c$$

$t = \log_3 x \quad dt = \frac{1}{x} \cdot \frac{1}{\ln 3} dx \quad \ln 3 dt = \frac{1}{x} dx$

Example 6

Evaluate the following integral

$$\int \frac{1}{e^{2x} + e^{-2x}} dx \quad 21 \text{ March } 2007 \text{ A}$$

Solution

$$I = \int \frac{1}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x}}{e^{4x} + 1} dx \quad \text{بالضرب في } e^{2x} \text{ بسط ومقام}$$

$\text{Let } t = e^{2x} \quad dt = 2e^{2x} dx \quad \frac{1}{2} dt = e^{2x} dx$

$$I = \frac{1}{2} \int \frac{1}{t^2 + 1} dt = \frac{1}{2} \tan^{-1}(t) + c = \frac{1}{2} \tan^{-1}(e^{2x}) + c$$

Example 7

Evaluate the integral

$$\int \frac{2^x}{1 + 4^x} dx \quad 20 \text{ November } 2006 \text{ A}$$

Solution

$$I = \int \frac{2^x}{1 + 4^x} dx = \int \frac{2^x}{1 + (2^x)^2} dx$$

$\text{Let } t = 2^x \quad dt = 2^x \ln 2 dx \quad \frac{1}{\ln 2} dt = 2^x dx$

$$I = \frac{1}{\ln 2} \int \frac{1}{t^2 + 1} dt = \frac{1}{\ln 2} \tan^{-1}(t) + c = \frac{1}{\ln 2} \tan^{-1}(2^x) + c$$



Example 8

Evaluate the following integrals

$$\int \frac{x^3 + x}{x^4 + 1} dx$$

10 March 1999

Solution

$$I = \int \frac{x^3 + x}{x^4 + 1} dx = \int \frac{x^3}{x^4 + 1} dx + \int \frac{x}{x^4 + 1} dx$$

$$I_1 = \int \frac{x^3}{x^4 + 1} dx \rightarrow \text{Let } t = x^4 + 1 \rightarrow dt = 4x^3 dx \rightarrow \frac{1}{4} dt = x^3 dx$$

$$I_1 = \frac{1}{4} \int \frac{1}{t} dt = \frac{1}{4} \ln|t| + c_1 = \frac{1}{4} \ln(x^4 + 1) + c_1$$

$$I_2 = \int \frac{x}{x^4 + 1} dx \rightarrow \text{Let } u = x^2 \rightarrow du = 2x dx \rightarrow \frac{1}{2} du = x dx$$

$$I_2 = \frac{1}{2} \int \frac{1}{u^2 + 1} du = \frac{1}{2} \tan^{-1} u + c_2 = \frac{1}{2} \tan^{-1} x^2 + c_2$$

$$I = I_1 + I_2 = \frac{1}{4} \ln(x^4 + 1) + \frac{1}{2} \tan^{-1} x^2 + c$$

Example 9

Evaluate the following integral

$$\int \frac{dx}{5x^{\frac{1}{2}} + x^{\frac{3}{2}}}$$

18 July 2005 A

Solution

$$I = \int \frac{dx}{5x^{\frac{1}{2}} + x^{\frac{3}{2}}} = \int \frac{dx}{x^{\frac{1}{2}}(5 + x)}$$

$$\text{Let } t = x^{\frac{1}{2}} \rightarrow du = \frac{1}{2} x^{-\frac{1}{2}} dx \rightarrow 2du = \frac{1}{x^{\frac{1}{2}}} dx$$

$$I = 2 \int \frac{1}{5 + t} dt = 2 \cdot \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{t}{\sqrt{5}} \right) + c = \frac{2}{\sqrt{5}} \tan^{-1} \left(\frac{x^{\frac{1}{2}}}{\sqrt{5}} \right) + c$$

Example 10

Evaluate the integral

$$\int \left(\frac{2e^x}{4 + e^{2x}} + \frac{e^{2x}}{4 + e^{2x}} \right) dx$$

28 April 2009 A

Solution

$$I = \int \left(\frac{2e^x}{4 + e^{2x}} + \frac{e^{2x}}{4 + e^{2x}} \right) dx = \int \frac{2e^x}{4 + e^{2x}} dx + \int \frac{e^{2x}}{4 + e^{2x}} dx$$

$$I_1 = \int \frac{2e^x}{4 + e^{2x}} dx \quad \text{Let } t = e^x \rightarrow dt = e^x dx$$

$$I_1 = 2 \int \frac{1}{4 + t^2} dt = 2 \cdot \frac{1}{2} \tan^{-1} \frac{t}{2} + c_1 = \tan^{-1} \left(\frac{e^x}{2} \right) + c_1$$

$$I_2 = \int \frac{e^{2x}}{4 + e^{2x}} dx \quad \text{Let } u = 4 + e^{2x} \rightarrow du = 2e^{2x} dx \rightarrow \frac{1}{2} du = e^{2x} dx$$

$$I_2 = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + c_2 = \frac{1}{2} \ln|4 + e^{2x}| + c_2$$

$$I = I_1 + I_2 = \tan^{-1} \left(\frac{e^x}{2} \right) + \frac{1}{2} \ln(4 + e^{2x}) + c$$



Example 11

Evaluate the following integral

$$\int \frac{1}{x \sqrt{9x^4 - 16}} dx \quad 21 \text{ March } 2007 \text{ A}$$

Solution

$$I = \int \frac{1}{x \sqrt{9x^4 - 16}} dx = \int \frac{x}{x^2 \sqrt{9x^4 - 16}} dx$$

$$t = x^2 \rightarrow dt = 2x dx \rightarrow \frac{1}{2} dt = x dx$$

$$I = \frac{1}{2} \int \frac{1}{t \sqrt{9t^2 - 16}} dt = \frac{1}{6} \int \frac{1}{t \sqrt{t^2 - \frac{16}{9}}} dt = \frac{1}{6} \cdot \frac{3}{4} \sec^{-1} \left(\frac{3t}{4} \right) + c = \frac{1}{8} \sec^{-1} \left(\frac{3x^2}{4} \right) + c$$

Example 12

Evaluate the following integral

$$\int \frac{1}{x \sqrt{x^3 - 4}} dx \quad 12 \text{ July } 2000 \text{ A}$$

Solution

$$I = \int \frac{1}{x \sqrt{x^3 - 4}} dx = \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{2}} \sqrt{x^3 - 4}} dx$$

$$\text{Let } t = x^{\frac{3}{2}} \rightarrow dt = \frac{3}{2} x^{\frac{1}{2}} dx \rightarrow \frac{2}{3} dt = x^{\frac{1}{2}} dx$$

$$I = \frac{2}{3} \int \frac{1}{t \sqrt{t^2 - 1}} dt = \frac{2}{3} \sec^{-1} t + c = \frac{2}{3} \sec^{-1} x^{\frac{3}{2}} + c$$

Example 13

Evaluate the following integral

$$\int \frac{dx}{\sqrt{e^{2x} - 1}} \quad 29 \text{ July } 2009 \text{ A}$$

Solution

$$I = \int \frac{dx}{\sqrt{e^{2x} - 1}} = \int \frac{e^x}{e^x \sqrt{e^{2x} - 1}} dx$$

$$\text{Let } t = e^x \rightarrow dt = e^x dx$$

$$I = \int \frac{1}{t \sqrt{t^2 - 1}} dt = \sec^{-1} t + c = \sec^{-1}(e^x) + c$$

Example 14

Evaluate the following integral

$$\int \frac{dx}{\sqrt{9^x - 4}} \quad 28 \text{ April } 2009 \text{ A}$$

Solution

$$I = \int \frac{dx}{\sqrt{9^x - 4}} = \int \frac{dx}{\sqrt{3^{2x} - 4}} = \int \frac{3^x}{3^x \sqrt{3^{2x} - 4}} dx$$

$$\text{Let } t = 3^x \rightarrow dt = 3^x \ln 3 dx \rightarrow \frac{1}{\ln 3} dt = 3^x dx$$

$$I = \frac{1}{\ln 3} \int \frac{1}{t \sqrt{t^2 - 4}} dx = \frac{1}{2 \ln 3} \sec^{-1} \left(\frac{t}{2} \right) + c = \frac{1}{2 \ln 3} \sec^{-1} \left(\frac{3^x}{2} \right) + c$$



Homework

<u>1</u>	Evaluate the following integrals	$\int \frac{dx}{(\sec^{-1} x) x \sqrt{x^2 - 1}}$	October 1996
<u>2</u>	Evaluate the following integrals	$\int \frac{dx}{(\csc^{-1} x) x \sqrt{x^2 - 1}}$	
<u>3</u>	Evaluate the following integrals	$\int \frac{dx}{(\sin^{-1} x) \sqrt{1 - x^2}}$	
<u>4</u>	Evaluate the following integrals	$\int \frac{dx}{(\cos^{-1} x) \sqrt{1 - x^2}}$	
<u>5</u>	Evaluate the integral	$\int \frac{1}{(x^2 + 1) \cot^{-1} x} dx$	
<u>6</u>	Evaluate the following integrals	$\int_0^1 \frac{x}{\sqrt{4 - x^4}} dx$	July 1997
<u>7</u>	Evaluate the integral	$\int \frac{dx}{(\csc x) \sqrt{4 - \cos^2 x}}$	27 November 2008 A



Homework

<u>8</u>	Evaluate the following integral	$\int \frac{1}{x\sqrt{1-(\ln x)^2}} dx$	4 July 1996
<u>9</u>	Evaluate the following integrals	$\int \frac{\cos x}{1+\sin^2 x} dx$	October 1997
<u>10</u>	Evaluate the following integral	$\int \frac{1}{x(9+(\ln x)^2)} dx$	24 March 2008 A
<u>11</u>	Evaluate the following integral	$\int \frac{e^{3-x}}{1+e^{-2x}} dx$	12 July 2000 A
<u>12</u>	Evaluate the following integral	$\int \frac{2x+1}{4x^2+1} dx$	29 July 2009 A
<u>13</u>	Evaluate the following integrals	$\int \frac{dx}{x\sqrt{x^8-9}} dx$	October 1997
<u>14</u>	Evaluate the following integral	$\int \frac{dx}{x\sqrt{x^8-4}}$	13 March 2001 A
<u>15</u>	Evaluate the following integrals	$\int \frac{1}{\sqrt{e^{2x}-4}} dx$	10 March 1999 11 October 1999
<u>16</u>	Evaluate the following integral	$\int \frac{1}{\sqrt{e^{2x}-9}} dx$	14 March 2002
<u>17</u>	Evaluate the following	$\int \frac{10^x}{100^x+81} dx$	[3 mark] 31 10 July 2010
<u>18</u>	Evaluate the following	$\int \frac{t+\sqrt{9-t^2}}{9-t^2} dt$	[3 mark] 31 10 July 2010
<u>19</u>	Evaluate the following	$\int \frac{dx}{\sqrt{4^x-1}} dx$	[3pts.] 32 Oct. 31 st , 2010 A

